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Kinematics of a UR5

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Contents

1 Introduction

This worksheet describes how to derive the forward and inverse kinematic equations of a UR5 robot. The worksheet is inspired by [Hawkins, 2013], [Keating, 2017], and [Kebria et al., 2016] but attempts to explain each step more thoroughly.

1.1 Notation

The worksheet follows the Denavit-Hartenberg notation used by [Craig, 2005], sometimes referred to as *modified* DH-parameters. Additionally, the following brief notations are used:

- ${}^0P_6 = \begin{bmatrix} {}^0P_{6x} \\ {}^0P_{6y} \\ {}^0P_{6z} \end{bmatrix}$ is the origin of frame 6 seen from frame 0.
- ${}^0\hat{Y}_6 = \begin{bmatrix} {}^0\hat{Y}_{6x} \\ {}^0\hat{Y}_{6y} \\ {}^0\hat{Y}_{6z} \end{bmatrix}$ is a unit vector giving the direction of the y-axis of frame 6 seen from frame 0.

- ${}^0_6T$ is a transformation from frame 6 to frame 0, meaning that ${}^0P = {}^0_6T \cdot {}^6P$.

2 Forward Kinematics for UR5

The forward kinematic (FK) equations calculates a transformation matrix ${}^0_6T$ based on known joint angles θ_{1-6}. The transformation matrix is defined as:

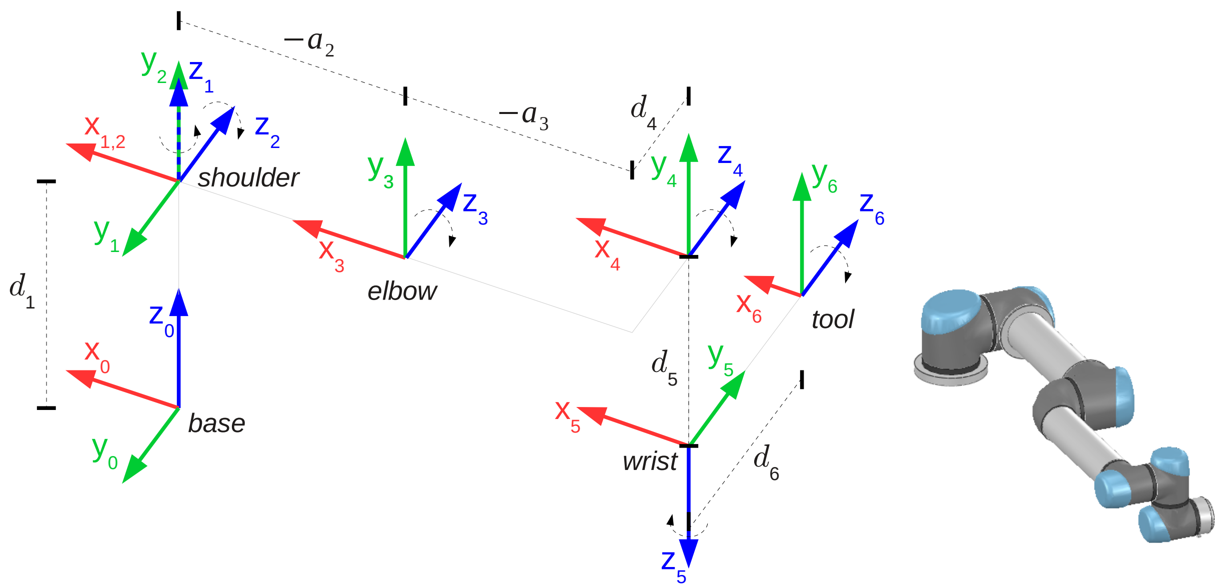
$$
\begin{aligned}
{}^0_6T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6) &= \begin{bmatrix} {}^0_6R & {}^0P_6 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} {}^0\hat{X}_6 & {}^0\hat{Y}_6 & {}^0\hat{Z}_6 & {}^0P_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} {}^0\hat{X}_{6x} & {}^0\hat{Y}_{6x} & {}^0\hat{Z}_{6x} & {}^0P_{6x} \\ {}^0\hat{X}_{6y} & {}^0\hat{Y}_{6y} & {}^0\hat{Z}_{6y} & {}^0P_{6y} \\ {}^0\hat{X}_{6z} & {}^0\hat{Y}_{6z} & {}^0\hat{Z}_{6z} & {}^0P_{6z} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{1}
$$

, where the columns ${}^0\hat{X}_6$, ${}^0\hat{Y}_6$, and ${}^0\hat{Z}_6$ are unit vectors defining the axes of frame 6 in relation to frame 0.

We can split the transformation matrix is given as a chain of transformations; one for each joint:

$$
{}^0_6T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6) = {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6) \tag{2}
$$

The kinematic structure of the UR5 robot in zero position is shown in Figure 1:



(a) *Frames describing the UR5 robot.* **(b)** *Visualization*

Figure 1: *The UR5 robot in zero position.*

The DH-parameters are according to [Craig, 2005] specified as:

$$\begin{aligned} a_i &= \text{distance from } Z_i \text{ to } Z_{i+1} \text{ measured along } X_i \\ \alpha_i &= \text{angle from } Z_i \text{ to } Z_{i+1} \text{ measured about } X_i \\ d_i &= \text{distance from } X_{i-1} \text{ to } X_i \text{ measured along } Z_i \\ \theta_i &= \text{angle from } X_{i-1} \text{ to } X_i \text{ measured about } Z_i \end{aligned} \tag{3}$$

For the UR5, the DH parameters are:

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	θ_1
2	$\alpha_1 = 90°$	0	0	θ_2
3	0	a_2	0	θ_3
4	0	a_3	d_4	θ_4
5	$\alpha_4 = 90°$	0	d_5	θ_5
6	$\alpha_5 = -90°$	0	d_6	θ_6

Table 1: *Modified Denavit-Harteberg parameters (DH-parameters) of a UR5 robot, corresponding to the frames in Figure 1. The parameters θ_i are variables and the remaining parameters are constants.*

The DH-parameters can be used to write the transformations for each link. The general transformation between link $i-1$ and i is given by[1]:

$$^{\text{i-1}}_{\text{i}}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos(\alpha_{i-1}) & \cos\theta_i\cos(\alpha_{i-1}) & -\sin(\alpha_{i-1}) & -\sin(\alpha_{i-1})d_i \\ \sin\theta_i\sin(\alpha_{i-1}) & \cos\theta_i\sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) & \cos(\alpha_{i-1})d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

It is straightforward to write the transformation matrix for each link of the UR5 robot by inserting the DH-parameters from Table 1 in Equation (4). The complete transformation from base to end-effector can then be derived by multiplication of all 6 transformation matrices, as shown in Equation (2). The result is analytical expressions for all 12 parameters in the transformation matrix ^0_6T. The complete analytic equations can be found in [Hawkins, 2013].

3 Inverse Kinematics for UR5

The inverse kinematic (IK) equations calculates the joint angles θ_{1-6} based on a desired position and orientation of the final frame, specified as the transformation ^0_6T. In the solution, we restrict all angles as $(\theta_1, ..., \theta_6) \in [0; 2\pi[$.

[1]Equation 3.6 in [Craig, 2005]

3.1 Finding θ_1

To find θ_1, we first determine the location of frame 5 (the *wrist* frame) in relation to the base frame; ${}^0P_5$. As illustrated in Figure 2, ${}^0P_5$ can be found by translating backwards from frame 6 to frame 5 along z_6. Remember that both ${}^0_6T$ and d_6 are known.

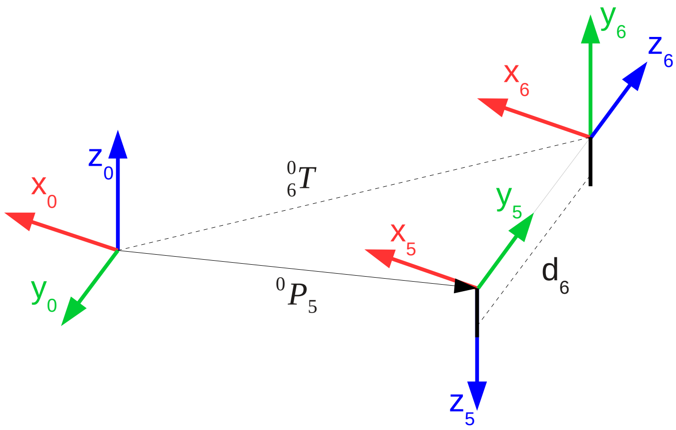


Figure 2: *Finding the origin of frame 5.*

The translation ${}^0P_5$ can be written as:

$$
\begin{aligned}
{}^0P_5 &= {}^0P_6 - d_6 \cdot {}^0\hat{Z}_6 \Leftrightarrow \\
{}^0P_5 &= {}^0_6T \begin{bmatrix} 0 \\ 0 \\ -d_6 \\ 1 \end{bmatrix}
\end{aligned}
\tag{5}
$$

To derive θ_1, we examine the robot seen from above in Figure 3 (looking down into z_0):

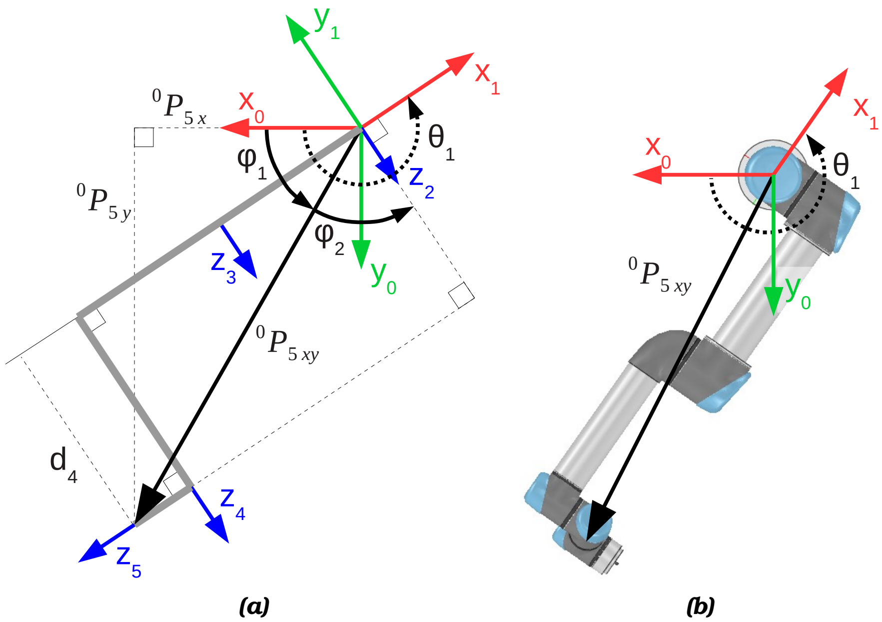


Figure 3: *Robot (until frame 5) seen from above. The robot is shown as grey lines. Note that z_5 should actually point into the page if all angles are 0. Here $\theta_4 \neq 0$ in order to better illustrate how θ_1 can be isolated.*

Our approach to finding θ_1 is to consider the wrist, P_5, seen from frame 0 and frame 1, respectively. Intuitively, the rotation from frame 0-to-1, θ_1, should equal the difference between the rotations from 0-to-5 and 1-to-5. Formalizing and inserting symbols from Figure 3 give:

$$\begin{aligned} v_{0\to1} &= v_{0\to5} - v_{1\to5} \Leftrightarrow \\ v_{0\to1} &= v_{0\to5} + v_{5\to1} \Leftrightarrow \\ \theta_1 &= \phi_1 + \left(\phi_2 + \frac{\pi}{2}\right) \end{aligned} \tag{6}$$

The angle ϕ_1 can be found by examining the triangle with sides ${}^0P_{5x}$ and ${}^0P_{5y}$:

$$\phi_1 = \text{atan2}({}^0P_{5y}, {}^0P_{5x}) \tag{7}$$

The angle ϕ_2 is found by examining the rightmost triangle with ϕ_2 as one of the angles. Two of the sides have lengths $|{}^0P_{5xy}|$ and d_4:

$$\begin{aligned} \cos(\phi_2) &= \frac{d_4}{|{}^0P_{5xy}|} \Rightarrow \\ \phi_2 &= \pm\text{acos}\left(\frac{d_4}{|{}^0P_{5xy}|}\right) \Leftrightarrow \\ \phi_2 &= \pm\text{acos}\left(\frac{d_4}{\sqrt{{}^0P_{5x}{}^2 + {}^0P_{5y}{}^2}}\right) \end{aligned} \tag{8}$$

The desired angle θ_1 can now be found simply as:

$$\theta_1 = \phi_1 + \phi_2 + \frac{\pi}{2} \Leftrightarrow$$

$$\theta_1 = \text{atan2}\left({}^0P_{5y}, {}^0P_{5x}\right) \pm \text{acos}\left(\frac{d_4}{\sqrt{{}^0P_{5x}{}^2 + {}^0P_{5y}{}^2}}\right) + \frac{\pi}{2} \tag{9}$$

The two solutions correspond to the shoulder being "left" or "right".

3.2 Finding θ_5

Figure 4 shows the robot from above again; this time with frame 6 included.

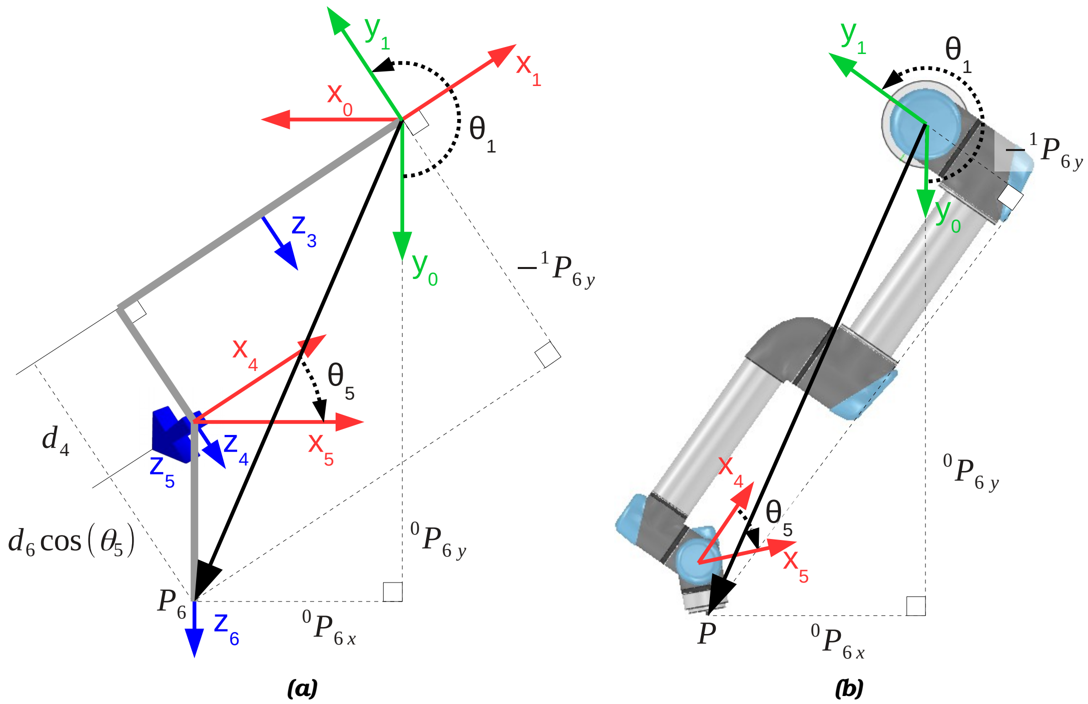


Figure 4: *Robot (including frame 6) seen from above.*

Our approach to finding θ_5 is to notice that ${}^1P_{6y}$ (the y-component of ${}^1P_6$) *only* depends on θ_5. In the figure, we can trace y_1 backwards to see that ${}^1P_{6y}$ is given by:

$$-{}^1P_{6y} = d_4 + d_6 \cos\theta_5 \tag{10}$$

The component ${}^1P_{6y}$ can also be expressed by looking at ${}^1P_6$ as a rotation of ${}^0P_6$

around z_1:

$$
\begin{aligned}
{}^{0}P_6 &= {}^{0}_{1}R \cdot {}^{1}P_6 \Leftrightarrow \\
{}^{1}P_6 &= {}^{0}_{1}R^{\top} \cdot {}^{0}P_6 \Leftrightarrow \\
\begin{bmatrix} {}^{1}P_{6x} \\ {}^{1}P_{6y} \\ {}^{1}P_{6z} \end{bmatrix} &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\top} \begin{bmatrix} {}^{0}P_{6x} \\ {}^{0}P_{6y} \\ {}^{0}P_{6z} \end{bmatrix} \Leftrightarrow \\
\begin{bmatrix} {}^{1}P_{6x} \\ {}^{1}P_{6y} \\ {}^{1}P_{6z} \end{bmatrix} &= \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{0}P_{6x} \\ {}^{0}P_{6y} \\ {}^{0}P_{6z} \end{bmatrix} \Rightarrow \\
{}^{1}P_{6y} &= {}^{0}P_{6x} \cdot (-\sin\theta_1) + {}^{0}P_{6y} \cdot \cos\theta_1
\end{aligned} \tag{11}
$$

By combining Equation (10) and (11), we can eliminate ${}^{1}P_{6y}$ and express θ_5 only using known values:

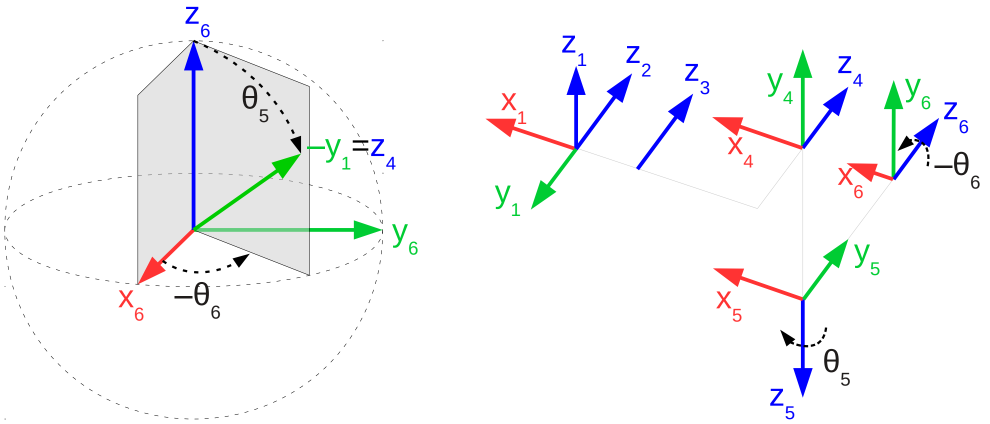
$$
\begin{aligned}
-d_4 - d_6 \cos\theta_5 &= {}^{0}P_{6x}(-\sin\theta_1) + {}^{0}P_{6y}\cos\theta_1 \Leftrightarrow \\
\cos\theta_5 &= \frac{{}^{0}P_{6x}\sin\theta_1 - {}^{0}P_{6y}\cos\theta_1 - d_4}{d_6} \Leftrightarrow \\
\theta_5 &= \pm\text{acos}\left(\frac{{}^{0}P_{6x}\sin\theta_1 - {}^{0}P_{6y}\cos\theta_1 - d_4}{d_6}\right)
\end{aligned} \tag{12}
$$

Again there are two solutions. These correspond to the wrist being “up” or “down”, respectively. This can be interpreted intuitively: The joint sum $(\theta_2 + \theta_3 + \theta_4)$ can cause the end-effector to be located in the same position, but with the wrist flipped. The orientation can then be “corrected” by θ_6.

Also note that a solution is defined as long as the value inside acos has a magnitude not greater than 1; equivalent to $|{}^{1}P_{6y} - d_4| \leq |d_6|$.

3.3 Finding θ_6

To determine θ_6 we examine y_1 seen from frame 6; ${}^{6}\hat{Y}_1$. This axis will (ignoring translations) always be parallel to ${}^{6}\hat{Z}_{2,3,4}$, as can be seen from Figure 5b. Therefore, it will only depend on θ_5 and θ_6. It turns out that $-{}^{6}\hat{Y}_1$ can in fact be described using spherical coordinates, where azimuth is $-\theta_6$ and the polar angle is θ_5; see Figure 5a.



***(a)** Polar angles for $-{}^{6}\hat{Y}_1$* ***(b)** Reference view of the relevant frames*

***Figure 5:** The axis $-{}^{6}\hat{Y}_1$ expressed in spherical coordinates with azimuth $-\theta_6$ and polar angle θ_5. For simplicity, ${}^{6}\hat{Y}_1$ is denoted y_1 in the Figure.*

Converting $-{}^{6}\hat{Y}_1$ from spherical to Cartesian coordinates gives:

$$-{}^{6}\hat{Y}_1 = \begin{bmatrix} \sin\theta_5\cos(-\theta_6) \\ \sin\theta_5\sin(-\theta_6) \\ \cos\theta_5 \end{bmatrix} \Leftrightarrow$$

$${}^{6}\hat{Y}_1 = \begin{bmatrix} -\sin\theta_5\cos\theta_6 \\ \sin\theta_5\sin\theta_6 \\ -\cos\theta_5 \end{bmatrix} \tag{13}$$

In Equation (13) we could isolate θ_6 and have an expression of θ_6 in relation to ${}^{6}_{1}T$. We want an expression of θ_6 all the way from ${}^{6}_{0}T$. To get this, we identify that ${}^{6}\hat{Y}_1$ is given as a rotation of θ_1 in the x, y-plane of frame 0 (very similar to Equation (11)):

$${}^{6}\hat{Y}_1 = {}^{6}\hat{X}_0 \cdot (-\sin\theta_1) + {}^{6}\hat{Y}_0 \cdot \cos\theta_1 \Leftrightarrow$$

$${}^{6}\hat{Y}_1 = \begin{bmatrix} -{}^{6}\hat{X}_{0x} \cdot \sin\theta_1 + {}^{6}\hat{Y}_{0x} \cdot \cos\theta_1 \\ -{}^{6}\hat{X}_{0y} \cdot \sin\theta_1 + {}^{6}\hat{Y}_{0y} \cdot \cos\theta_1 \\ -{}^{6}\hat{X}_{0z} \cdot \sin\theta_1 + {}^{6}\hat{Y}_{0z} \cdot \cos\theta_1 \end{bmatrix} \tag{14}$$

Equating the first two entries of (13) and (14) give:

$$\left.\begin{aligned} -\sin\theta_5\cos\theta_6 &= -{}^{6}\hat{X}_{0x} \cdot \sin\theta_1 + {}^{6}\hat{Y}_{0x} \cdot \cos\theta_1 \\ \sin\theta_5\sin\theta_6 &= -{}^{6}\hat{X}_{0y} \cdot \sin\theta_1 + {}^{6}\hat{Y}_{0y} \cdot \cos\theta_1 \end{aligned}\right\} \Leftrightarrow \tag{15}$$

$$\left\{\begin{aligned} \cos\theta_6 &= \tfrac{{}^{6}\hat{X}_{0x} \cdot \sin\theta_1 - {}^{6}\hat{Y}_{0x} \cdot \cos\theta_1}{\sin\theta_5} \\ \sin\theta_6 &= \tfrac{-{}^{6}\hat{X}_{0y} \cdot \sin\theta_1 + {}^{6}\hat{Y}_{0y} \cdot \cos\theta_1}{\sin\theta_5} \end{aligned}\right\} \Rightarrow$$

$$\theta_6 = \operatorname{atan2}\left(\frac{-{}^{6}\hat{X}_{0y} \cdot \sin\theta_1 + {}^{6}\hat{Y}_{0y} \cdot \cos\theta_1}{\sin\theta_5}, \frac{{}^{6}\hat{X}_{0x} \cdot \sin\theta_1 - {}^{6}\hat{Y}_{0x} \cdot \cos\theta_1}{\sin\theta_5} \right) \tag{16}$$

This solution is undetermined if the denominator $\sin\theta_5 = 0$. In this case, the joint axes 2, 3, 4 and 6 are aligned (as in Figure 5b). This is “too many” degrees of freedom.

The axes 2, 3, and 4 can on their own rotate the end-effector (frame 6) around z_6 without moving it, and the 6'th joint therefore becomes redundant. In this case, θ_6 can simply be set to an arbitrary value.

If both of the numerators in Equation (16) are 0, the solution is *also* undetermined. If this is the case, $\sin\theta_5$ must also be 0, and the situation is thus the same. This can be seen by examining both sides in Equation (15).

3.4 Finding θ_3

We examine the remaining three joints (2, 3, and 4). Notice that their joint axes are all parallel. Together they constitute a planar 3R-manipulator, as illustrated in Figure 6.

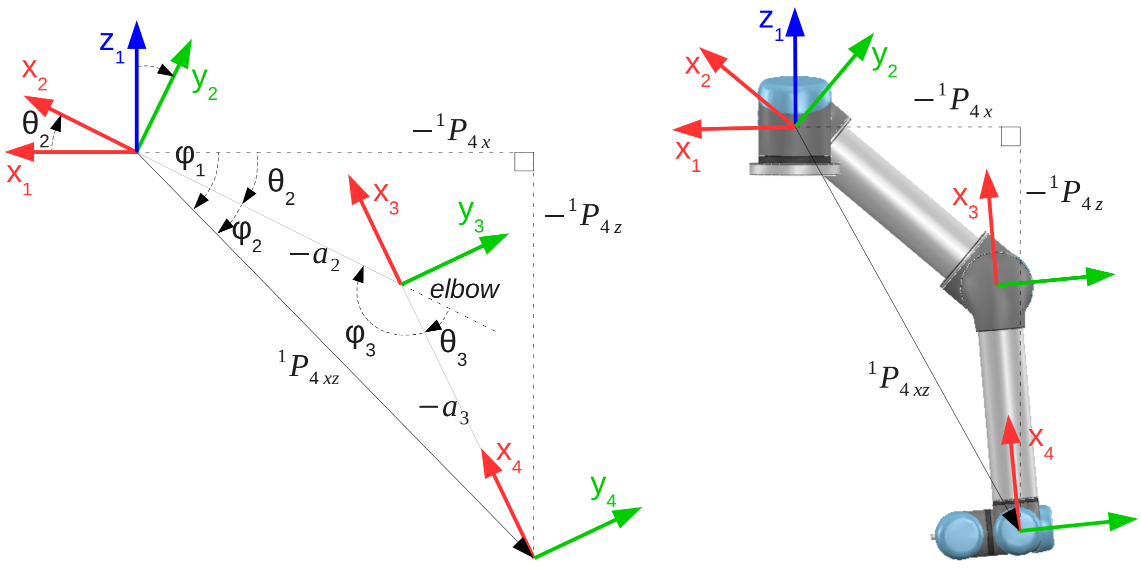


Figure 6: *Joint 2, 3, and 4 together constitutes a 3R planar manipulator.*

We can constrict ourselves to look at ${}^1_4T$ (frame 4 in relation to frame 1) because ${}^0_1T$, ${}^4_5T$, and ${}^5_6T$ at this point are known. This transformation is illustrated in the x,z-plane of frame 1 in Figure 6a. From the figure it is clear that the length of the translation $|{}^1P_{4xz}|$ is determined only by θ_3, or similarly by ϕ_3. The angle ϕ_3 can be found by using the law of cosine:

$$\cos\phi_3 = \frac{(-a_2)^2 + (-a_3)^2 - |{}^1P_{4xz}|^2}{2(-a_2)(-a_3)} = \frac{{a_2}^2 + {a_3}^2 - |{}^1P_{4xz}|^2}{2a_2a_3} \tag{17}$$

The relationship between $\cos\phi_3$ and $\cos\theta_3$ is:

$$\cos\theta_3 = \cos(\pi - \phi_3) = -\cos(\phi_3) \tag{18}$$

Combining (17) and (18) give:

$$\cos\theta_3 = -\frac{{a_2}^2 + {a_3}^2 - |\,^1P_{4xz}|^2}{2a_2a_3} \Leftrightarrow$$
$$\theta_3 = \pm \text{acos}\left(\frac{|\,^1P_{4xz}|^2 - {a_2}^2 - {a_3}^2}{2a_2a_3}\right) \tag{19}$$

Note that solutions exist for θ_3 if the argument of acos is within $[-1;1]$. It can be shown that this is equivalent to $|\,^1P_{4xz}| \in [|a_2 - a_3|; |a_2 + a_3|]$. In most cases where solutions exist, there will exist two different solutions. These correspond to "elbow up" and "elbow down".

3.5 Finding θ_2

The angle θ_2 can be found as $\phi_1 - \phi_2$. Each of these can be found by inspecting Figure 6a and using atan2 and sine relations:

$$\phi_1 = \text{atan2}(-\,^1P_{4z}, -\,^1P_{4x}) \tag{20}$$
$$\frac{\sin\phi_2}{-a_3} = \frac{\sin\phi_3}{|\,^1P_{4xz}|} \Leftrightarrow$$
$$\phi_2 = \text{asin}\left(\frac{-a_3\sin\phi_3}{|\,^1P_{4xz}|}\right) \tag{21}$$

We can replace ϕ_3 with θ_3 by noticing that $\sin\phi_3 = \sin(180° - \theta_3) = \sin\theta_3$. Combining the equations now give:

$$\theta_2 = \phi_1 - \phi_2 = \text{atan2}(-\,^1P_{4z}, -\,^1P_{4x}) - \text{asin}\left(\frac{-a_3\sin\theta_3}{|\,^1P_{4xz}|}\right) \tag{22}$$

3.6 Finding θ_4

The last remaining angle θ_4 is defined as the angle from X_3 to X_4 measured about Z_4 (c.f. Equation (3) in page 3). It can thus easily be derived from the last remaining transformation matrix, ${}^3_4T$, using its first column ${}^3\hat{X}_4$:

$$\theta_4 = \text{atan2}({}^3\hat{X}_{4y}, {}^3\hat{X}_{4x}) \tag{23}$$

4 Discussion

To sum up, a total of 8 solutions exist in general for the general inverse kinematic problem of the UR5: $2_{\theta_1} \times 2_{\theta_5} \times 1_{\theta_6} \times 2_{\theta_3} \times 1_{\theta_2} \times 1_{\theta_4}$.

4.1 Additional Material

Each of the sources used as inspiration for this document contain more information in specific subjects. Most notably: [Hawkins, 2013], [Keating, 2017], and [Kebria et al., 2016]

- *Full FK solution* is included in [Hawkins, 2013].

- *Dynamics* is briefly covered in [Kebria et al., 2016].
- *An extra axis* is added in [Hawkins, 2013].
- *Multiple IK solutions* are visualized in [Keating, 2017]

5 References

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